Mode Orthogonality in Reciprocal and Nonreciprocal Waveguides

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***Abstract*—Using a general reciprocity theorem as a basis, the orthogonality relations for lossy reciprocal and nonreciprocal waveguides are discussed. To obtain a useful orthogonality relation which can extract a particular mode from a general mode expansion, a reciprocal waveguide must be bidirectional. A nonreciprocal waveguide, however, must be mutually bidirectional with its complementary waveguide (obtained by reversing the dc magnetic field applied to the gyrotropic media). For these bidirectionality conditions to be met, a waveguide must possess at least one of three symmetries: reflection, 180° rotation or rotary reflection symmetry. In those cases warranted by the structure symmetry, simplified forms for the orthogonality relations are presented. The orthogonality relations for the special case of lossless reciprocal or nonreciprocal waveguides are also discussed.**

I. Introduction

Modes provide a convenient basis for describing the electromagnetic behavior of transmission lines, waveguides, optical fibers, etc. [1]–[4]. In order for it to be "easy" to represent the electromagnetic fields by a mode expansion, the modes should be orthogonal. This paper discusses useful orthogonality relations for lossy reciprocal and nonreciprocal waveguides based on a general reciprocity theorem. Attention is restricted to structures containing, at most, gyrotropic media. Although the discussion could be extended to structures containing bianisotropic media [5], they will not be covered here.

Only uniform waveguides, invariant to a translation parallel to their axis, are considered. Because of this translation symmetry, the basis functions for the electromagnetic field components have the form: $f_m(x, y)e^{-\gamma_m z}$, where γ_m is a complex number ($e^{j\omega t}$ time variation is assumed). The modes form either an infinite discrete spectrum (closed boundary structures), or a finite discrete spectrum plus a continuous spectrum (open boundary structures). In general, each mode will include two branches, corresponding to propagation in the $+z(\gamma_m^+)$ and $-z(\gamma_m^-)$ directions. An important consideration in the discussion of the orthogonality of modes is whether $\gamma_m^- = -\gamma_m^+$; or more generally, whether there are pairs of modes such that $\gamma_n^- = -\gamma_m^+$. Waveguides which satisfy this condition are termed bidirectional [6].

Section II discusses the mode orthogonality relation derived from a reciprocity theorem [3], [7]–[9] that applies to both reciprocal and nonreciprocal waveguides. In order for this mode orthogonality relation to be useful in developing mode expansions for an arbitrary excitation, the waveguide must be bidirectional. A bidirectional waveguide must possess at least one of three symmetries relative to its axis (the z-axis): reflection in a plane perpendicular to the z-axis, 180° rotation about an axis perpendicular to the z-axis, or rotary reflection symmetry about the z-axis. Most waveguides of current interest for microwave applications possess reflection and/or 180° rotation symmetry.

Section III discusses the mode orthogonality relations derived from the reciprocity theorem for lossy waveguides containing either isotropic media, or anisotropic media whose susceptibilities are symmetric tensors; i.e., reciprocal waveguides. Section IV discusses the mode orthogonality relations derived for lossy waveguides containing gyromagnetic media; i.e., nonreciprocal waveguides. Any gyromagnetic medium used will, in general, have a dc magnetic field, H_o, applied, producing a dc magnetization, M_o, parallel to H_o. To develop the mode orthogonality relations, it is necessary to simultaneously consider the complementary waveguide obtained by reversing the direction of H_o (and, therefore, the direction of M_o). One finds that the orthogonality relations operate between modes of the original and the complementary waveguide. Although only waveguides containing gyromagnetic media are examined, the extension of the results to waveguides containing gyroelectric media can be easily made.

The discussion in Sections II–IV applies to waveguides in general, including those containing dissipative materials. Waveguides containing only lossless media have special properties with the consequence that the propagating modes are bidirectional regardless of the presence of one of the spatial symmetries listed above. The implications of this for the orthogonality of the modes in lossless waveguides is discussed in Section V.

II. General Reciprocity Theorem

A general statement of the reciprocity principle applied to the modes of a uniform waveguide of infinite length is

Manuscript received May 2, 1990; revised June 11, 1991.
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IEEE Log Number 9102811

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(see, for example, section 6.1 of [3]):

$$(\gamma_n' + \gamma_m) \iint [e_n'(x,y) \times \boldsymbol{h}_m(x,y) - \boldsymbol{e}_m(x,y) \times \boldsymbol{h}_n'(x,y)] \cdot \boldsymbol{a}_z \, dx\,dy = 0. \quad (1)$$

The integral is over the cross section of the waveguide, $\boldsymbol{e}_m(x,y)e^{-\gamma_m z}, \boldsymbol{h}_m(x,y)e^{-\gamma_m z}$ are the electric and magnetic fields for the mth mode of the waveguide, and $\boldsymbol{a}_z$ is a unit vector parallel to the waveguide axis. For a reciprocal waveguide, the primed fields refer to the nth mode of the waveguide. For a nonreciprocal waveguide containing gyromagnetic media, the primed fields refer to the nth mode of the complementary waveguide. The original and complementary waveguides are identical except that the dc magnetic field (and the dc magnetization) are reversed in the complementary waveguide.

Equation (1), a general orthogonality relation for either reciprocal or nonreciprocal waveguides, states that the integral must be zero if $\gamma_n' \neq -\gamma_m$. Thus, if the unprimed and primed fields belong to the same branches of their respective modes, then the integral is zero because γ_n' and γ_m cannot sum to zero. The integral in (1) can be nonzero only if the unprimed and primed fields belong to different branches of their respective modes and $\gamma_n^{-\prime} = -\gamma_m^+$. In a bidirectional waveguide, for each mode m there will exist a mode p with $\gamma_p = -\gamma_m$ for all ω. For each such mode pair the integral in (1) may be nonzero and thereby provide a means of extracting a particular mode from a mode expansion. Otherwise, if $(\gamma_n' + \gamma_m)$ is never zero, then the integral must equal zero for all pairs of modes, and (1) can provide no procedure to evaluate the contribution of a particular mode in an expansion. Thus, the successful application of the general orthogonality relation (1) to evaluate the contribution of a particular mode to a mode expansion for a waveguide depends on the waveguide being bidirectional.

A sufficient condition for a waveguide to be bidirectional is that the structure possess at least one symmetry operation which interchanges the + and − z axes. After applying such a symmetry operation, the "reversed" structure appears identical with the original, and its mode spectrum is identical with the original mode spectrum. Looking in the $+z$ direction of the reversed waveguide is equivalent to looking in the $-z$ direction of the original waveguide. As a consequence, the waveguide will have pairs of modes with $\gamma_p^- = -\gamma_m^+, \gamma_p^+ = -\gamma_m^-$. All waveguides which contain only isotropic media are bidirectional, and for them, $\gamma_m^- = -\gamma_m^+$. However, for some bidirectional waveguides which contain anisotropic media, $p \neq m$; an example is the sheath helix [10]–[12].

There are three symmetry operations that interchange the + and $-z$ axes: reflection in a plane perpendicular to the z-axis, rotation by 180° about an axis perpendicular to the z-axis, or rotary reflection about the z-axis. Rotary reflection involves rotation by π/n radians about the z-axis, followed by reflection in a plane perpendicular to the z-axis. Any uniform waveguide possessing at least one of these symmetry operations must be bidirectional.[1]

III. Mode Orthogonality in Reciprocal Waveguides

This section considers bidirectional reciprocal waveguides. These are waveguides that contain isotropic media, or anisotropic media whose dielectric and magnetic susceptibilities are symmetric tensors, and that, in addition, possess at least one of the three symmetry operations listed above.

Consider a mode of a waveguide propagating in the $+z$ direction $(\boldsymbol{e}_n^+, \boldsymbol{h}_n^+)$ with propagation constant γ_n^+. If a symmetry operation S which interchanges the + and $-z$ axes is applied to the waveguide, then $(e_n^+, \boldsymbol{h}_n^+)$ is still a solution because the reversed waveguide is identical to the original. Let $P(S)$ be a symmetry operation on the modal electromagnetic field (structure fixed) which produces the same electromagnetic field-structure relation that the S operation on the structure (electromagnetic field fixed) yields. The resulting electromagnetic field is labeled $(e_n^{-\prime}, h_n^{-\prime})$ with propagation constant $\gamma_n^{-\prime}$; this mode is labeled with a superscript − because it travels in the $-z$ direction.

$$\boldsymbol{e}_n^{-\prime}(x,y)e^{-\gamma_n^{-\prime} z} = P(S)\left[e_n^+(x,y)e^{-\gamma_n^+ z}\right],$$
$$\boldsymbol{h}_n^{-\prime}(x,y)e^{-\gamma_n^{-\prime} z} = P(S)\left[h_n^+(x,y)e^{-\gamma_n^+ z}\right]. \quad (2)$$

Because the symmetry operation reverses the direction of travel of the mode, $\gamma_n^{-\prime} = -\gamma_n^+$. The mode with a prime traveling in the $-z$ direction is the image, produced by $P(S)$, of the unprimed mode traveling in the $+z$ direction. This transformed mode $(e_n^{-\prime}, h_n^{-\prime})$ need not be identical in form to the original mode $(\boldsymbol{e}_n^+, \boldsymbol{h}_n^+)$ because the symmetry operation may rearrange the field components. However, the transformed mode must be similar to one of the modes of the original structure because the transformation is based on a symmetry operation which leaves the structure unchanged. These transformed modes can be used in (1) because they are also modes of the original structure.

An alternative form of (1) is given (the primes are omitted since only reciprocal waveguides are considered). Let $p(m)$ label that mode for which $\gamma_{p(m)}^- = -\gamma_m^+$, and the superscript σ denote + or −. The orthogonality relation for closed boundary waveguides, expressed in terms of the transverse fields of the modes (denoted by the subscript T), is

$$\iint \left[\boldsymbol{e}_{Tp(n)}^{\sigma} \times \boldsymbol{h}_{Tm}^{+} - e_{Tm}^{+} \times h_{Tp(n)}^{\sigma}\right] \cdot a_z \, dx\,dy = N_m^{+-} \delta_{p(n)m}[1 - \delta_{\sigma+}], \quad (3a)$$

$$\iint \left[\boldsymbol{e}_{Tp(n)}^{\sigma} \times \boldsymbol{h}_{Tm}^{-} - e_{Tm}^{-} \times h_{Tp(n)}^{\sigma}\right] \cdot a_z \, dx\,dy = N_m^{-+} \delta_{p(n)m}[1 - \delta_{\sigma-}]. \quad (3b)$$

[1] In a previous paper concerning bidirectionality in gyrotropic waveguides [6], it was stated that reciprocal waveguides are bidirectional. That statement is incorrect; reciprocity is not a sufficient condition for bidirectionality in a lossy waveguide.

The δ functions are defined by: $\delta_{p(n)m}=0$, $n \neq m$, and $\delta_{p(m)m}=1$. Also, $\delta_{+-}=0$, and $\delta_{++}=\delta_{--}=1$. The normalization constants, N_m^{+-} and N_m^{-+}, are the values of the integrals in (3a) and (3b), respectively, when $p(n)=p(m)$. Note that the normalization constants, which involve the only nonzero values of the integrals in (1) or (3), require different branches for the two modes, m and $p(m)$.

For open boundary waveguides, the $\delta_{p(n)m}$ on the right side of 3(a)–(b) should be replaced by $\delta(p(n)-m)$ when the continuous spectrum is involved; see [13]. Although the methods of this analysis apply to open boundary as well as to closed boundary structures, for the sake of brevity the subsequent discussion will refer explicitly to the latter case.

The mode expansion for the transverse fields in a reciprocal bidirectional waveguide is

$$E_T(x,y,z) = \sum_n \left[A_n e_{Tn}^+(x,y) e^{-\gamma_n^+ z} + B_n e_{Tn}^-(x,y) e^{-\gamma_n^- z}\right], \quad (4a)$$

$$H_T(x,y,z) = \sum_n \left[A_n h_{Tn}^+(x,y) e^{-\gamma_n^+ z} + B_n h_{Tn}^-(x,y) e^{-\gamma_n^- z}\right]. \quad (4b)$$

Using (3), one finds that the mode coefficients are

$$A_m e^{-\gamma_m^+ z} = \frac{1}{N_m^{+-}} \iint \left[e_{Tp(m)}^-(x,y) \times H_T(x,y,z) - E_T(x,y,z) \times h_{Tp(m)}^-(x,y)\right] \cdot a_z \, dx\, dy, \quad (5a)$$

$$B_m e^{-\gamma_m^- z} = \frac{1}{N_m^{-+}} \iint \left[e_{Tp(m)}^+(x,y) \times H_T(x,y,z) - E_T(x,y,z) \times h_{Tp(m)}^+(x,y)\right] \cdot a_z \, dx\, dy. \quad (5b)$$

A. Reflection Symmetry

Equations (3a) and (3b) and (5a) and (5b) can be simplified somewhat for each symmetry type. This has been done previously only for reflection symmetry [1]–[4]; this case is summarized first. Let M denote reflection in a plane perpendicular to the z-axis. When applying $P(M)$ to the mode fields, one must recognize that the electric field is a polar vector and the magnetic field is an axial vector (see section 6.11 of [14] for a discussion of polar and axial vectors). For reflection symmetry, (2) yields

$$e_n^{-\prime} e^{-\gamma_n^{-\prime} z} = \left[e_{Tn}^+ - a_z e_{zn}^+\right] e^{\gamma_n^+ z},$$
$$h_n^{-\prime} e^{-\gamma_n^{-\prime} z} = -\left[h_{Tn}^+ - a_z h_{zn}^+\right] e^{\gamma_n^+ z}. \quad (6)$$

For waveguides with reflection symmetry, (6) implies that $\gamma_n^- = -\gamma_n^+$, $e_{Tn}^- = e_{Tn}^+ = e_{Tn}$, $h_{Tn}^- = -h_{Tn}^+ = -h_{Tn}$. The fields of the image mode traveling in the $-z$ direction, produced by the reflection operation acting on the original mode traveling in the $+z$ direction, can be expressed in terms of the fields of the original mode by inserting appropriate minus signs. Combining these results with (3), the orthogonality relation for reciprocal waveguides with reflection symmetry is

$$2\iint \left[e_{Tn}(x,y) \times h_{Tm}(x,y)\right] \cdot a_z \, dx\, dy = N_m^{+-}\delta_{nm} = -N_m^{-+}\delta_{nm}. \quad (7)$$

Equation (7) applies to waveguides containing isotropic media, to waveguides containing an uniaxial dielectric (if the optical axis of the permittivity tensor is either parallel, or perpendicular, to the z-axis), and to waveguides containing a biaxial dielectric (if one of the principal axes of the permittivity tensor is parallel to the z-axis) [15]. Setting $N_m^{+-} = -N_m^{-+} = N_m$, the expressions in (5a) and (5b) for the coefficients in the mode expansion of (4a) and (4b) can be simplified to

$$A_m e^{-\gamma_m z} = \frac{1}{N_m} \iint \left[e_{Tm}(x,y) \times H_T(x,y,z) + E_T(x,y,z) \times h_{Tm}(x,y)\right] \cdot a_z \, dx\, dy, \quad (8a)$$

$$B_m e^{+\gamma_m z} = \frac{-1}{N_m} \iint \left[e_{Tm}(x,y) \times H_T(x,y,z) - E_T(x,y,z) \times h_{Tm}(x,y)\right] \cdot a_z \, dx\, dy. \quad (8b)$$

These are typical of the expressions given in the previous literature [1]–[4].

B. 180° Rotation Symmetry

Next, waveguides for which a 180° rotation about an axis perpendicular to the z-axis is the only symmetry operation are considered (denote this rotation by R and set $\theta = 0$ at the R axis). The fields for the + branch of the mth mode of a waveguide with this symmetry can be written in the form:

$$e_m^+(r,\theta) e^{-\gamma_m^+ z} = \sum_{l=-\infty}^{\infty} \left[a_r e_{rml}^+(r) + a_\theta e_{\theta ml}^+(r) + a_z e_{zml}^+(r)\right] e^{jl\theta} e^{-\gamma_m^+ z}, \quad (9a)$$

$$h_m^+(r,\theta) e^{-\gamma_m^+ z} = \sum_{l=-\infty}^{\infty} \left[a_r h_{rml}^+(r) + a_\theta h_{\theta ml}^+(r) + a_z h_{zml}^+(r)\right] e^{jl\theta} e^{-\gamma_m^+ z}. \quad (9b)$$

Because of the 180° rotation symmetry, for each mode m there must be a corresponding mode $p(m)$ such that $\gamma_{p(m)}^- = -\gamma_m^+$. Modes m and $p(m)$ are distinct and form a bidirectional pair. The fields of mode $p(m)$ are

$$e_{p(m)}^-(r,\theta) e^{-\gamma_{p(m)}^- z} = \sum_{l=-\infty}^{\infty} \left[a_r e_{rml}^+(r) - a_\theta e_{\theta ml}^+(r) - a_z e_{zml}^+(r)\right] e^{-jl\theta} e^{+\gamma_m^+ z}, \quad (10a)$$

$$h_{p(m)}^-(r,\theta) e^{-\gamma_{p(m)}^- z} = \sum_{l=-\infty}^{\infty} \left[a_r h_{rml}^+(r) - a_\theta h_{\theta ml}^+(r) - a_z h_{zml}^+(r)\right] e^{-jl\theta} e^{+\gamma_m^+ z}. \quad (10b)$$

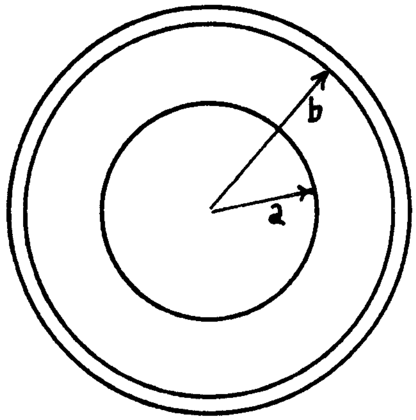


Fig. 1. Sheath helix (radius a) with concentric shield (radius b).

The normalization integrals of 3(a) and 3(b) can be written in this case as (R_o is the radius which bounds the structure):

$$N_m^{+-}=2\pi\sum_{l=-\infty}^{\infty}\int_0^{R_o}\left[e_{rm,-l}^{+}(r)h_{\theta ml}^{+}(r)\right.$$
$$+e_{\theta m,-l}^{+}(r)h_{rml}^{+}(r)+e_{rml}^{+}(r)h_{\theta m,-l}^{+}(r)$$
$$\left.+e_{\theta ml}^{+}(r)h_{rm,-l}^{+}(r)\right]rdr, \qquad (11a)$$

$$N_m^{-+}=2\pi\sum_{l=-\infty}^{\infty}\int_0^{R_o}\left[e_{rm,-l}^{-}(r)h_{\theta ml}^{-}(r)\right.$$
$$+e_{\theta m,-l}^{-}(r)h_{rml}^{-}(r)+e_{rml}^{-}(r)h_{\theta m,-l}^{-}(r)$$
$$\left.+e_{\theta ml}^{-}(r)h_{rm,-l}^{-}(r)\right]rdr. \qquad (11b)$$

Thus the coefficients in a mode expansion are given by

$$A_me^{-\gamma_m^+z}=\frac{1}{N_m^{+-}}\int_0^{2\pi}\int_0^{R_o}\left[e_{Tp(m)}^{-}(r,\theta)\times H_T(r,\theta,z)\right.$$
$$\left.-E_T(r,\theta,z)\times \boldsymbol{h}_{Tp(m)}^{-}(r,\theta)\right]\cdot \boldsymbol{a}_z r\,dr\,d\theta, \qquad (12a)$$

$$B_me^{-\gamma_m^-z}=\frac{1}{N_m^{-+}}\int_0^{2\pi}\int_0^{R_o}\left[e_{Tp(m)}^{+}(r,\theta)\times H_T(r,\theta,z)\right.$$
$$\left.-E_T(r,\theta,z)\times \boldsymbol{h}_{Tp(m)}^{+}(r,\theta)\right]\cdot \boldsymbol{a}_z r\,dr\,d\theta. \qquad (12b)$$

One example of a waveguide with this symmetry is a circular sheath helix [10]–[12]. For a sheath helix surrounded by a concentric shield, Fig. 1, one finds that the summation in (9a) and (9b) can be dropped. Denoting the radial variation by the subscript q, each combination of the integers l and q ($-\infty\leqslant l\leqslant\infty$, $1\leqslant q\leqslant\infty$) identifies a distinct mode. Each of these modes has two branches with propagation constants γ_{lq}^+ and γ_{lq}^-. For this structure $\gamma_{lq}^-\neq-\gamma_{lq}^+$ if $l\neq0$. However, the distinct modes, $\{l,q\}$ and $\{-l,q\}$, form a bidirectional pair with $\gamma_{-lq}^-=\gamma_{lq}^+$. The normalization constants are

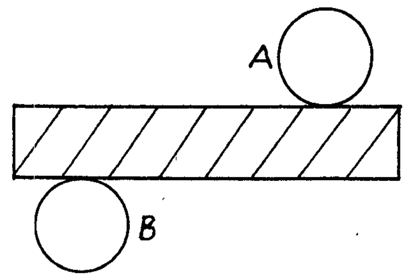


Fig. 2. Right-handed (A) and left-handed (B) sheath helices of equal and opposite pitch angle supported by an isotropic dielectric slab.

$$N_{lq}^{+-}=4\pi\int_0^b\left[e_{rlq}^{+}(r)h_{\theta lq}^{+}(r)+e_{\theta lq}^{+}(r)h_{rlq}^{+}(r)\right]rdr, \qquad (13a)$$

$$N_{lq}^{-+}=4\pi\int_0^b\left[e_{rlq}^{-}(r)h_{\theta lq}^{-}(r)+e_{\theta lq}^{-}(r)h_{rlq}^{-}(r)\right]rdr. \qquad (13b)$$

Equations (12a) and (12b) give the mode coefficients where one should identify mode m with mode $\{l,q\}$ and mode $p(m)$ with mode $\{-l,q\}$ for the sheath helix.

C. Rotary Reflection Symmetry

The third class of waveguides is one for which the structures are invariant to a rotation of π/n radians about the z-axis followed by a reflection in a plane perpendicular to the z-axis (denote this symmetry operation by S_{2n}). Waveguides possessing this symmetry (but having neither of the symmetries discussed previously) are not important for current microwave applications. An example of a waveguide with this symmetry ($n=1$) is a dielectric slab supporting two sheath helices (one right-handed and the other left-handed, with equal and opposite pitch angles) on opposite faces of the slab, as shown in Fig. 2.

Following a procedure similar to that for the 180° rotation symmetry case, one finds that the waveguide is bidirectional with $\gamma_{p(m)}^-=-\gamma_m^+$. In general for waveguides with rotary reflection symmetry, $\gamma_m^-\neq-\gamma_m^+$. The normalization integrals of (3a) and (3b) can be written in this case as

$$N_m^{+-}=2\pi\sum_{l=-\infty}^{\infty}e^{jl\pi/n}\int\left[e_{rm,-l}^{+}(r)h_{\theta ml}^{+}(r)\right.$$
$$-e_{\theta m,-l}^{+}(r)h_{rml}^{+}(r)+e_{rml}^{+}(r)h_{\theta m,-l}^{+}(r)$$
$$\left.-e_{\theta ml}^{+}(r)h_{rm,-l}^{+}(r)\right]rdr, \qquad (14)$$

with a similar expression for N_m^{-+} (the + superscripts replaced by − superscripts). The coefficients in a mode expansion are again given by expressions of the type shown in (12a) and (12b), but using (14) for the normalization constants.

IV. Mode Orthogonality in Nonreciprocal Waveguides

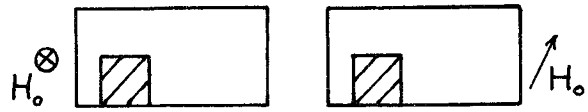
This section examines waveguides containing gyromagnetic media; for example, ferrite material with a dc magnetic field applied. If the dc magnetic field, H_o, and the resulting dc magnetization of the ferrite, M_o, are in the $+z$ direction, the ac permeability is (section 4.1 of [16]):

$$\boldsymbol{\mu}(\omega,H_o,M_o) = \mu_o \begin{bmatrix} \mu(\omega,H_o,M_o) & -j\kappa(\omega,H_o,M_o) & 0 \\ j\kappa(\omega,H_o,M_o) & \mu(\omega,H_o,M_o) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (15)$$

If the direction of the dc magnetic field (and, therefore, the dc magnetization) is reversed, the permeability is $\boldsymbol{\mu}(\omega,-H_o,-M_o)=\boldsymbol{\mu}^{\dagger}(\omega,H_o,M_o)$, where the $\boldsymbol{\mu}^{\dagger}$ indicates the transpose of $\boldsymbol{\mu}$ [17]. Introduction of the dc magnetic field, $\boldsymbol{H}_o$, establishes its direction as a unique direction for the waveguide. The components of the ac magnetic field perpendicular to the dc magnetic field will be strongly influenced by the presence of the ferrite material. For example, the dispersion characteristics of two modes with ac magnetic field components which are CW and CCW elliptically polarized relative to the direction of H_o will differ significantly; see chapter 9 of [16]. Another detailed exposition of propagation in a variety of waveguides containing ferrites is given in [18].

To apply the orthogonality relation given in (1) to a waveguide containing ferrite material, it is necessary to consider at the same time the complementary waveguide obtained by reversing the direction of the dc magnetic field applied to the ferrite in the waveguide. Equation (1) states that the integral involving mode m of the original waveguide and mode n of the complementary waveguide must be zero unless $\gamma_n' = -\gamma_m$. Thus, to obtain a nonzero value for this integral the modes of the original and complementary waveguides must be able to be grouped in pairs such that $\gamma_{p(m)'}^{-} = -\gamma_m^{+}$. In this case, the development of a useful orthogonality relation requires that the waveguide and its complementary waveguide be mutually bidirectional.

The original and complementary waveguides differ only in the directions of their dc magnetic fields, $\boldsymbol{H}_o$ and $\boldsymbol{H}_o' = -\boldsymbol{H}_o$. Thus, a symmetry operation on the original waveguide will produce a waveguide which is also identical with the complementary waveguide (except, perhaps, for a change in the direction of the dc magnetic field). Now, however, when applying a symmetry operation to the mode fields (structure fixed) we must examine whether the resulting orientation of the ac magnetic fields relative to the direction of the dc magnetic field of the complementary waveguide is the same as in the original waveguide, or whether it has been reversed. See [19] for a discussion of the symmetry properties of fields in waveguides with gyrotropic media.



(a) (b)

Fig. 3. Ferrite-loaded rectangular waveguide with: (a) axial dc magnetic field and (b) transverse dc magnetic field.

Let $p(m)$ label that mode of the complementary waveguide for which $\gamma_{p(m)'}^{-} = -\gamma_m^{+}$, where γ_m^{+} is the propagation constant (+branch) of mode m of the original waveguide. Then (1) can be rewritten as

$$\iint \left[\boldsymbol{e}_{Tp(n)'}^{\sigma} \times \boldsymbol{h}_{Tm}^{+} - \boldsymbol{e}_{Tm}^{+} \times \boldsymbol{h}_{Tp(n)'}^{\sigma} \right] \cdot \boldsymbol{a}_z \, dx\,dy = N_m^{+-} \delta_{p(n)m} \left[1 - \delta_{\sigma+}\right], \quad (16a)$$

$$\iint \left[\boldsymbol{e}_{Tp(n)'}^{\sigma} \times \boldsymbol{h}_{Tm}^{-} - \boldsymbol{e}_{Tm}^{-} \times \boldsymbol{h}_{Tp(n)'}^{\sigma} \right] \cdot \boldsymbol{a}_z \, dx\,dy = N_m^{-+} \delta_{p(n)m} \left[1 - \delta_{\sigma-}\right]. \quad (16b)$$

The normalization constants, N_m^{+-} and N_m^{-+}, involve different branches for the two modes; mode m of the original waveguide, and mode $p(m)$ of the complementary waveguide.

A. Reflection Symmetry

Examples of ferrite-loaded waveguides with reflection symmetry are shown in Fig. 3. When applying the reflection operator, M, to these structures, one must take into account that the dc magnetic field, $\boldsymbol{H}_o$, is an axial vector. After applying M to the structure of Fig. 3(a) (axial dc magnetic field), the resulting image structure looks exactly like the original. On the other hand, after applying M to the structure of Fig. 3(b) (transverse dc magnetic field), the resulting image structure looks like the complementary waveguide (Fig. 3(b) with $\boldsymbol{H}_o$ reversed). Regardless of the direction of the dc magnetic field, applying the reflection operator, $P(M)$, to the nth mode of the original waveguide yields the result given in (6) above. Note that the ac magnetic field components transverse to the z direction are reversed, while the z component is not (a consequence of the magnetic field being an axial vector).

Consider first the case when the dc magnetic field H_o is in the $+z$ direction (Fig. 3(a)). Applying the reflection operator M to this structure reverses the direction of the z-axis, but does not reverse the direction of H_o. Thus in the coordinates for the reflected structure, H_o points in the $-z$ direction. Consider a specific mode of the original waveguide propagating in the $+z$ direction, say mode m with ac magnetic field components, $\boldsymbol{H}_{TRm}$, perpendicular to the dc magnetic field. For reflection symmetry, mode m of the original structure (propagating in the $+z$ direction) is a valid mode for the reflected structure, propagating in the $-z$ direction of the reflected structure. If the reflection operator $P(M)$ is applied to the fields of mode m (structure fixed), by symmetry the reflected mode is a valid mode of the original waveguide. This reflected mode travels in the $-z$ direction, and its transverse ac magnetic field components, $\boldsymbol{H}_{TRm'}$, are reversed relative to H_{TRm}. This reversal, however, does not change the polarization of $\boldsymbol{H}_{TRm}$ relative to $\boldsymbol{H}_o$, and the influence of the ferrite medium on the original mode, and on the reflected mode, is the same. One concludes that each mode of the structure is bidirectional with two branches such that $\gamma_m^- = -\gamma_m^+$. Further, the electric and magnetic field components of the $-$ branch are essentially the same as those of the $+$ branch (the only change is in the relative signs of the transverse and z components).

Some simplification in the orthogonality relations can be made in this case. Recalling that (6) applies to the modes of either the original, or the complementary, waveguide, $\gamma_m^- = -\gamma_m^+ = -\gamma_m, \boldsymbol{e}_{Tn}^- = \boldsymbol{e}_{Tn}^+ = \boldsymbol{e}_{Tn}, \boldsymbol{h}_{Tn}^- = -\boldsymbol{h}_{Tn}^+ = -\boldsymbol{h}_{Tn}$, etc. These results enable (16a) and (16b) to be slightly simplified.

$$\iint\left[\boldsymbol{e}_{Tp(n)}(x,y)'\times\boldsymbol{h}_{Tm}(x,y)+\boldsymbol{e}_{Tm}(x,y)\times\boldsymbol{h}_{Tp(n)}(x,y)'\right]\cdot\boldsymbol{a}_z\,dx\,dy = N_m^{+-}\delta_{p(n)m} = -N_m^{-+}\delta_{p(n)m} = N_m\delta_{p(n)m}. \quad (17)$$

The coefficients which appear in the mode expansions (5a) and (5b) for the electric and magnetic fields are given by

$$A_m e^{-\gamma_m z} = \frac{1}{N_m}\iint\left[\boldsymbol{e}_{Tp(m)}(x,y)'\times H_T(x,y,z)+E_T(x,y,z)\times h_{Tp(m)}(x,y)'\right]\cdot\boldsymbol{a}_z\,dx\,dy, \quad (18a)$$

$$B_m e^{+\gamma_m z} = \frac{-1}{N_m}\iint\left[\boldsymbol{e}_{Tp(m)}(x,y)'\times H_T(x,y,z)-E_T(x,y,z)\times h_{Tp(m)}(x,y)'\right]\cdot\boldsymbol{a}_z\,dx\,dy. \quad (18b)$$

In these expressions, $\boldsymbol{e}_{Tp(m)}(x,y)'$ and $\boldsymbol{h}_{Tp(m)}(x,y)'$ are the transverse electric and magnetic field components of mode $p(m)$ of the complementary waveguide (dc magnetic field reversed); this mode is identified by $\gamma_{p(m)'} = \gamma_m$ of the original waveguide.

Consider now the case when the dc magnetic field H_o is transverse to the z direction (Fig. 3(b)) and set $\boldsymbol{H}_o = \boldsymbol{a}_x H_o$. Applying the reflection operator M to this structure reverses the direction of the z-axis and also reverses the direction of H_o. The resulting structure looks exactly like the complementary waveguide, although the z-axes of the reflected and the complementary waveguides point in opposite directions. Strictly, the reflection operator M by itself is not a symmetry operator for this structure; the symmetry operator must combine M with a reversal of the dc magnetic field (to recover the structure of Fig. 3(b)). Consider a specific mode of the original waveguide propagating in the $+z$ direction, say mode m with ac magnetic field components $H_{TRm} = \boldsymbol{a}_y H_{my} + \boldsymbol{a}_z H_{mz}$ perpendicular to H_o (not to the z-axis). If the reflection operator $P(M)$ is applied to the fields of mode m (structure fixed), the reflected mode is not a valid mode of the original waveguide because M (by itself) is not a symmetry operator. This reflected mode, however, has $H_{TRm'} = -\boldsymbol{a}_y H_{my} + \boldsymbol{a}_z \boldsymbol{H}_{mz}$. These transverse field components have the same relationship to the dc magnetic field of the complementary waveguide as the original transverse field components had to the dc magnetic field of the original waveguide. Thus this new mode, obtained by reflection of a mode of the original waveguide, is a valid mode of the complementary waveguide traveling in the $-z$ direction.

We conclude that waveguides of this type will not be self-bidirectional. That is, one should not expect that $\gamma_m^- = -\gamma_m^+$ except, possibly, for a subset of the total set of modes. However, the original and complementary waveguides are mutually bidirectional. Any mode m of the original waveguide traveling in the $+z$ direction has an image mode, $p(m)$, traveling in the $-z$ direction in the complementary waveguide, such that $\gamma_{p(m)'}^- = -\gamma_m^+$. The fields of this image mode, $p(m)$, are obtained by applying the operation $P(M)$ to the fields of the original mode m.

In this case there can be a significant simplification in the form of the orthogonality relations. Recognizing that

$$e_{Tp(n)'}^- = P(M)[e_{Tn}^+] = e_{Tn}^+,$$
$$h_{Tp(n)'}^- = P(M)[h_{Tn}^+] = -h_{Tn}^+,$$

etc., (16a) and (16b) can be written as

$$\iint\left[\boldsymbol{e}_{Tn}^+\times\boldsymbol{h}_{Tm}^+ + \boldsymbol{e}_{Tm}^+\times\boldsymbol{h}_{Tn}^+\right]\cdot\boldsymbol{a}_z\,dx\,dy = N_m^{+-}\delta_{nm}, \quad (19a)$$

$$\iint\left[\boldsymbol{e}_{Tn}^-\times\boldsymbol{h}_{Tm}^- + \boldsymbol{e}_{Tm}^-\times\boldsymbol{h}_{Tn}^-\right]\cdot\boldsymbol{a}_z\,dx\,dy = N_m^{-+}\delta_{nm}. \quad (19b)$$

Similarly, the coefficients which appear in the mode expansions (4a) and (4b) for the electric and magnetic fields are given by

$$A_m e^{-\gamma_m^+ z} = \frac{1}{N_m^{+-}}\iint\left[e_{Tm}^+(x,y)\times H_T(x,y,z)+\boldsymbol{E}_T(x,y,z)\times h_{Tm}^+(x,y)\right]\cdot a_z\,dx\,dy, \quad (20a)$$

$$B_m e^{-\gamma_m^- z} = \frac{1}{N_m^{-+}}\iint\left[e_{Tm}^-(x,y)\times H_T(x,y,z)+\boldsymbol{E}_T(x,y,z)\times h_{Tm}^-(x,y)\right]\cdot a_z\,dx\,dy. \quad (20b)$$

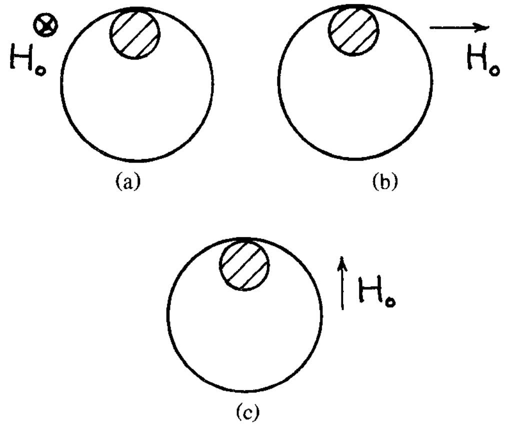


Fig. 4. Sheath helix containing a ferrite rod with: (a) axial dc magnetic field, and (b), (c) transverse dc magnetic field.

These expressions have been written in terms of the field components of the modes of the original waveguide, although the basic orthogonality relations involve modes of both the original and complementary waveguides. In this case, e_{Tm}^{+} and h_{Tm}^{+} are the transverse fields of the mth mode traveling in the $+z$ direction (+branch) of the original waveguide, while $e_{Tm}^{+}$and $-h_{Tm}^{+}$ are the transverse fields of the mth mode traveling in the $-z$ direction (−branch) of the complementary waveguide.

A waveguide with geometric reflection symmetry, but whose dc magnetic field is neither parallel to, nor perpendicular to, the waveguide axis, must also be considered. Now, neither a pure reflection M, nor a reflection M combined with a reversal of the dc magnetic field, is a symmetry operation for the waveguide. In general, it is not possible to find pairs of modes, m and $p(m)$ for the original and complementary waveguides, respectively, such that $\gamma_{p(m)'}^{-} = -\gamma_{m}^{+}$, and the original and complementary waveguides are not mutually bidirectional. Referring back to (1), the integral must be zero for (almost) all pairs of modes of the original and complementary waveguides. Thus, in this case (1) fails to provide a method to distinguish between modes in a mode expansion.

B. 180° Rotation Symmetry

Examples of ferrite-loaded waveguides with 180° rotation symmetry are shown in Fig. 4. Consider first the case when the dc magnetic field H_o is in the z direction (Fig. 4(a)). Applying the rotation operator R to this structure reverses the direction of the z-axis and also reverses the direction of $\boldsymbol{H}_o$. The resulting structure looks exactly like the complementary waveguide (Fig. 4(a) with $\boldsymbol{H}_o$ reversed). Strictly, the rotation operator R by itself is not a symmetry operator for the structure; the symmetry operator must combine R with a reversal of the dc magnetic field (to recover the structure of Fig. 4(a)). The application of $P(R)$ to the fields of a mode of the original structure yields a "rotated" mode which is not a valid mode of the original structure, but which is a valid mode of the complementary waveguide.

In general, waveguides of this type will not be self-bidirectional, but the original and complementary waveguides are mutually bidirectional. Any mode m of the original waveguide traveling in the $+z$ direction has an image mode, $p(m)$, traveling in the $-z$ direction in the complementary waveguide, such that $\gamma_{p(m)'}^{-} = -\gamma_{m}^{+}$, etc. The fields of this image mode, $p(m)$, are obtained by applying the operation $P(R)$ to the fields of the original mode m. These same conclusions apply to waveguides of the type shown in Fig. 4(b), and also to waveguides intermediate between those shown in Figs. 4(a) and 4(b). That is, they apply to waveguides where the 180° rotation axis of R is perpendicular to the dc magnetic field H_o.

In these cases there can be a significant simplification in the form of the orthogonality relations given in (16a) and (16b). The fields of the mth mode of the original waveguide can be written as in (9a) and (9b), and the fields of the image mode $p(m)$ can be written as in (10a) and (10b). The normalization integrals are given in (11a) and (11b), and the expressions for the mode coefficients in a mode expansion by (12a) and (12b). The only changes required in (10) through (12) are that a prime should be appended to any field component with a subscript $p(m)$; this labels these field components as belonging to the complementary waveguide with a reversed dc magnetic field.

Next, consider the case when the dc magnetic field $\boldsymbol{H}_o$ is parallel to the 180° rotation axis (Fig. 4(c)). After applying the rotation operator, R, to the structure of Fig. 4(c), the resulting image structure looks exactly like the original. Therefore, the modes of this structure must occur in pairs which are mutually bidirectional. What is the effect of reversing the direction of the dc magnetic field to obtain the complementary waveguide? Although the complementary waveguide is at least superficially similar to the original waveguide, there is no reason to expect that the modes of the complementary waveguide will be the same as the modes of the original waveguide. Thus, the original and complementary waveguides are not mutually bidirectional, and (1) may fail to provide a method to distinguish between modes in a mode expansion.

C. Rotary Reflection Symmetry

Ferrite-loaded waveguides with rotary reflection symmetry only are not important for microwave applications. An example of such a waveguide with S_2 symmetry is obtained from Fig. 2 by replacing the dielectric slab with a ferrite slab and applying a dc magnetic field. If the dc magnetic field is in the z direction, it is found that the original and complementary waveguides are, in general, not bidirectional; thus (1) may fail to provide a procedure to distinguish between modes. However, if the dc magnetic field is in the transverse plane, then the original and complementary waveguides are mutually bidirectional. In this case the results of Section III-C can be adapted to give the normalization constants and mode expansions.

V. Mode Orthogonality in Lossless Waveguides

Bidirectionality was ensured in the previous sections by the presence of a spatial symmetry. This symmetry is a sufficient, but not always necessary, condition. In a lossless structure, the propagating modes are bidirectional. Some aspects of the orthogonality of modes in lossless waveguides have been discussed in [2], [8], [20].

For a lossless reciprocal waveguide, the media permittivities and permeabilities are real; that is, $\epsilon = \boldsymbol{\epsilon}^*$, $\boldsymbol{\mu} = \boldsymbol{\mu}^*$. Let $e_m^+ e^{-\gamma_m^+ z}$ and $h_m^+ e^{-\gamma_m^+ z}$ be the electric and magnetic fields of the $+$ branch for mode m of a lossless reciprocal waveguide. Taking the complex conjugate of Maxwell's equations, one finds that $e_m^{+*} e^{-\gamma_m^{+*} z}$ and $-\boldsymbol{h}_m^{+*} e^{-\gamma_m^{+*} z}$ are the electric and magnetic fields of a valid mode of the structure; call this the mode conjugate to mode m. For a frequency ω at which mode m propagates, $\gamma_m^+ = j\beta_m^+$, and $\gamma_m^{+*} = -j\beta_m^+$, where β_m^+ is a real number. In this case the mode conjugate to m, a valid mode of the waveguide, travels in the $-z$ direction. In general, this conjugate mode is not the $-$branch of the original mode, but it is a distinct mode of the waveguide. Thus for propagating modes at least, a lossless reciprocal waveguide is bidirectional, regardless of the presence, or absence, of one of spatial symmetry operations (M, R, or S_{2n}) discussed previously.

For a frequency ω at which mode m is evanescent $\gamma_m^{+*} = \gamma_m^+$ because γ_m^+ is real. In this case the mode conjugate to mode m has exactly the same z dependence; it is, in fact, the same mode. No conclusions can be drawn about bidirectionality for such evanescent modes without considering the structure geometry, and it may not be possible to use (1) to isolate one of them. Finally, there are lossless waveguides which, over a finite frequency range, may have a pair of modes with propagation constants which are both complex, and are complex conjugates of each other [21]. In these cases, again, no conclusions can be drawn about the bidirectionality without considering the structure geometry.

In lossless nonreciprocal waveguides containing gyromagnetic media, the dyadic permeability $\boldsymbol{\mu}$ is Hermitian, and the parameters μ and κ which appear in it are real. Taking the complex conjugate of $\boldsymbol{\mu}$ is equivalent to taking its transpose; this, in turn, is equivalent to reversing the direction of the dc magnetic field. Thus, the mode which is conjugate to mode m of the original waveguide is a valid mode of the complementary waveguide (dc magnetic field reversed). If ω is such that mode m propagates in the $+z$ direction, then the conjugate mode propagates in the $-z$ direction in the complementary waveguide, and mutual bidirectionality is established. If ω is such that mode m is evanescent in the $+z$ direction, then the conjugate mode is also evanescent in the $+z$ direction (in the complementary waveguide), and mutual bidirectionality is not determined. So for nonreciprocal as well as reciprocal waveguides, losslessness provides the bidirectionality required toenable a propagating mode to be distinguished from any other propagating or evanescent mode in a mode expansion.

VI. Comments

The orthogonality relations presented above for lossy reciprocal waveguides with 180° rotation symmetry, or with rotary reflection symmetry, have not been previously discussed in the literature. Of course, a waveguide may possess two, or all three, of these symmetries simultaneously. Whenever more than one of these symmetries occurs, each of the corresponding orthogonality relations applies to the structure. If one of the symmetries is reflection, then (7) and (8a) and (8b) are the most efficient to use. The simultaneous presence of two, or all three, of these symmetries implies the presence of other symmetry operations about the waveguide axis. The total collection, or group, of symmetry operations for a waveguide controls the number of mode classes, and their degeneracies, for the waveguide [22].

There are, of course, lossy reciprocal and nonreciprocal waveguides which do not possess any of the three symmetry operations discussed above. In general such waveguides will not be bidirectional, and the reciprocity theorem fails to provide a means to isolate a particular mode from the other modes in a mode expansion.

Acknowledgment

The author thanks Professor Olle Nilsson and his colleagues at the Department of Microwave Engineering, the Royal Institute of Technology, Stockholm, Sweden, for their hospitality during a visit where a portion of this work was done.

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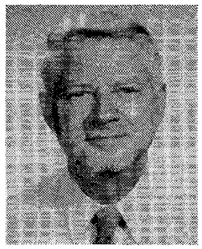

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